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UNITED STATES
EXPERIMENTAL MODEL BASIN

NAVY YARD, WASHINGTON, D.C.

THE CRITICAL EXTERNAL PRESSURE OF CYLINDRICAL TUBES UNDER UNIFORM RADIAL AND AXIAL LOAD

BY R. VON MISES

[redacted]

AUGUST 1933

REPORT 366
TR[illegible] 6

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by R. von Mises.

Translated and Annotated by D. F. Windenburg.

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THE CRITICAL EXTERNAL PRESSURE OF CYLINDRICAL

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by R. von Mises,

Berlin

Stodola's Festschrift, Zurich, 1929, pp. 418-430.

Translated and Annotated
by D. F. Windenburg*

The magnitude of buckling pressure and of the deformation of a cylindrical tube closed at both ends and subjected to external pressure is calculated on the basis of elasticity equations for thin shells. The results of the theory are compared with tests on two series of models. Complete agreement is shown with regard to the lobe formation; the problem of determining the buckling pressure has not yet been completely solved.

More then 14 years ago, in view of the conditions prevailing with regard to the design of fire tubes of boilers, I calculated the buckling pressure of cylindrical tubes which, with fixed ends, are subjected to external pressure. A somewhat different problem is presented when a circular cylinder, closed at the ends by means of bulkheads, is deeply submerged in water and thereby exposed to pressure from all sides, as is approximately the case of submarines. The changes in theory caused by the introduction of axial load are not very extensive, but the altered numerical ratios result in the occurrence of phenomena that necessarily have a certain intrinsic interest, especially concerning the large number of waves or bulges that becomes visible on the circumference of the cylinder at the moment of buckling; see Fig. 6, page 12.

In the following, I present a supplement to my previous publication# along the lines just indicated. The fundamental equations, which, by the way, are to be found in every text book on the theory of elasticity, and the detailed calculations carried out in the previous article, are not repeated; reference to that article is indicated by the symbol Z.

[Translator's Note: The translation of the previous publication, of which this is the supplement, is given in U. S. Experimental Model Basin Report No. 309. It should be referred to in connection with this article.]

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* [illegible] Physi[illegible], U. S. Experimental Model Basin, Washington, D. C.

[illegible] Ingen., 1914, p. 750.

1. DERIVATION of the FUNDAMENTAL FORMULA.

Let us consider a thin-walled, circular, cylindrical, hollow body, closed at the ends by flat heads, and subjected to the pressure p on its shell and to the compressive force P' on its heads; Fig. 1.

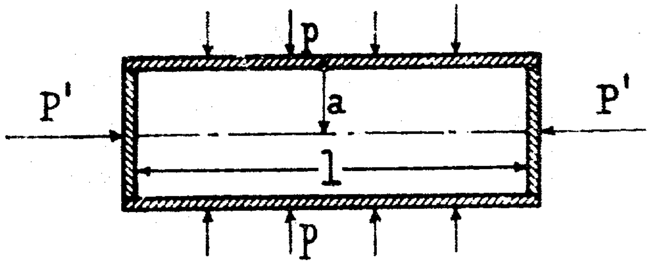


FIG. 1

Let the axial pressure on a unit cross-section of the circular ring having a diameter 2a and thickness 2h have a value p'. If the pressure on the heads is due to the fact that the end surfaces are subjected to the external pressure p, it follows from:

$P' = \pi a^2 p = \pi\, 2a\, 2h\, p'$

that $p' = p\, a/4h$ . . . . (1)

In order to take into consideration the presence of an end pressure p' the conditions of equilibrium of Z must be supplemented. For those equations are incomplete inasmuch as only terms of the first order in stress and deformation magnitudes are retained in them. Products of stresses and deformations of the second order of magnitude are omitted, with the exception of the products of p by w and $\frac{\partial^2 w}{\partial \varphi^2}$, (Z, Eq. (9)) since p, unlike all other stresses, is not of the order of magnitude of the deformations accompanying buckling but possesses a finite value independent of them. We must now, in an analogous manner, include also products of p' with deformation magnitudes in our equations.

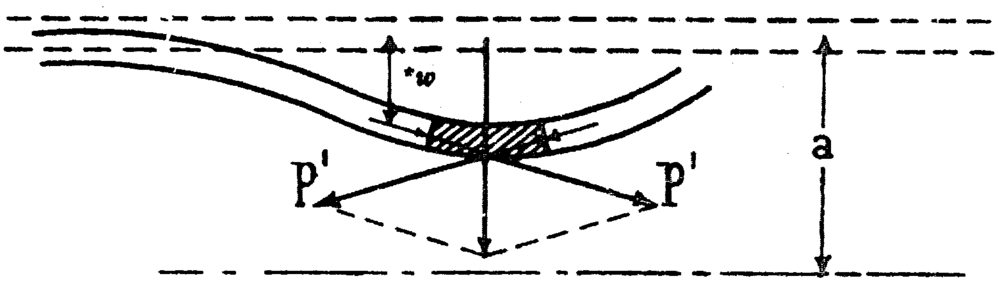


FIG. 2

Fig. 2 shows an element of the shell in a longitudinal plane through the axis of the tube, in the deformed condition. The generatrix, originally rectilinear and parallel to the x-axis, has acquired a curvature, which, except for terms of higher order, is measured by

$$\frac{\partial^2 w}{\partial x^2}$$

This expression, in the case of Fig. 2, is negative. It is seen that

the pressure p' 2h which acts on the surface of the cross-section of the element, produces a resultant in the w direction (radially inward, hence negative) equal to

$$-p' \, 2h \, \frac{\partial^2 w}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This expression, with opposite sign, must therefore be added on the right side of equation (9) in Z, if we would take into account a general end loading. Substituting the value of Eq. (1) in (2) we have:

$$-pa \, \tfrac{1}{2} \, \frac{\partial^2 w}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We must, therefore, instead of $-\dfrac{\partial^2 w}{\partial \varphi^2}$ in the parenthesis on the right hand side of Eq. (9) in Z, write

$$-\frac{\partial^2 w}{\partial \varphi^2} - \frac{a^2}{2} \frac{\partial^2 w}{\partial x^2}$$

The same remarks apply to Eq. (9') and (III) in Z. The end loading has no further influence on the expressions for the determination of the buckling pressure, as long as we remain within the limits of the simplifying assumptions introduced in Z or presupposed.

It is not difficult to follow the influence of the supplementary term, equation (3), through to the final formula in Z.

From Eq. (11) of Z and the abbreviation α given in Eq. (12) of Z, we have

$$a^2 \frac{\partial^2 w}{\partial x^2} = -\alpha^2 w, \qquad \alpha = \frac{\pi a}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

(l = effective length or frame spacing; see below, Section 2).

Therefore,

$$y\,(1 - n^2 - \alpha^2/2)$$

is to be substituted on the right hand side of Eq. (III') of Z in place of $y(1 - n^2)$; n represents the number of lobes. This same substitution is valid for the last member in the determinant (16) of Z. Since now y is the only quantity in the final equation that contains p, and therefore, is directly proportional to p, it follows that: <u>The buckling pressure of a tube, including the effect of end load, may be obtained by multiplying the pressure determined from Z by the factor</u>

$$\frac{n^2 - 1}{n^2 + \dfrac{\alpha^2}{2} - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This statement applies primarily to the complete result of the calculation carried out in Z, Eq. (A) or (A'), but it may be directly applied also to the simplified expression (B), since the assumptions that led to B -- smallness of

x and y defined by Eq. (14) of Z – – continue to hold generally. If we write (E) **somewhat differently (by setting the** expression in the parenthesis over a denominator and substituting partially for ρ its value from Eq. (18) of Z), we get from the foregoing expression:

$$y = \frac{1 - \sigma^2}{n^2 + \frac{\alpha^2}{2} - 1}\left(\frac{\alpha^2}{\alpha^2 + n^2}\right)^2 + \frac{x}{n^2 + \frac{\alpha^2}{2} - 1}\left[(n^2 + \alpha^2)^2 - 2\,\mu_1 n^2 + \mu_2\right]. \quad .(6)$$

where

$$\mu_1 = \tfrac{1}{2}\left[1 + (1 + \sigma)\rho\right]\left[2 + (1 - \sigma)\rho\right] = 1 + 1.65\rho + 0.455\rho^2$$

$$\mu_2 = (1 - \sigma\rho)\left[1 + (1 + 2\sigma)\rho - (1 - \sigma^2)(1 + \frac{1 + \sigma}{1 - \sigma}\rho)\,\rho^2\right]$$

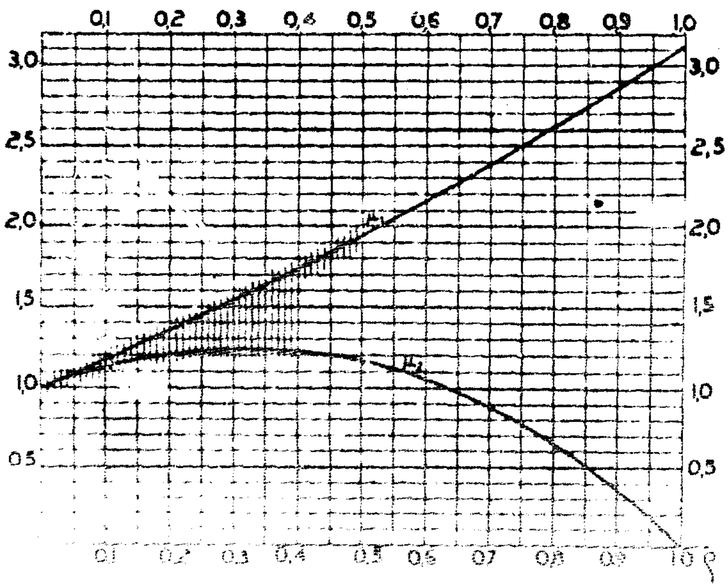
$$= 1 + 1.3\rho - 1.39\,\rho^2 - 1.417\,\rho^2 + 0.507\rho^4$$

Here as in Z

$$x = \frac{h^2}{3a^2}, \qquad y = p\,a/2h\,\frac{1 - \sigma^2}{E}, \qquad \rho = \frac{\alpha^2}{n^2 + \alpha^2} \quad . . . .(6')$$

and E and σ are the elastic constants ($\sigma = 0.3$). The determination of y from Eq. (6) is not particularly cumbersome once the two functions of ρ, μ_1 and μ_2, have been computed, and placed in the form of tables or curves for μ_1 and μ_2. Fig. 3 contains the two curves for the entire range $\rho = 0$ to 1 sufficiently adequate for all practical cases.

[Translator's Note: The error noted in equation (B) of Z which was carried through to the final formula (D) in Z has been corrected in the derivation of formula (6). Hence, formula (6) is correct despite the error in the original equation (B) in Z.]



[illegible] Auxiliary Functions μ_1 and μ_2 for [illegible] Computations by Formula (6).

The further simplification which was effected in Z by the assumption of only small values of ρ cannot be used here since the ratio of the radius to the frame spacing is sometimes too large. On the other hand, in most calculations of submarine hulls we have to do with a larger number of lobes, n. It is possible, therefore, if n equals at least 8 or 10, to neglect the last two terms in the brackets of Eq. (6) in comparison with the first term $(n^2 + \alpha^2)^2$. In the same way, we may neglect 1 in the denominators in comparison with n^2, by which the error made by the previous assumption is partly compensated. We have, then, with sufficiently close approximation for all cases of practical importance, the final formula:

$$y = \frac{1}{n^2 + \frac{\alpha^2}{2}} \left[\left(\frac{\alpha^2}{\alpha^2 + n^2} \right)^2 (1 - \sigma^2) + (n^2 + \alpha^2)^2 \, x \right] \quad \ldots \ldots \ldots \ldots \quad (7)$$

This is even simpler than the equations (C) and (D) in Z. If we substitute here as before $\sigma = 0.3$, and further for x and y the values from Eq. (6'), we obtain for the required buckling pressure p, if 2a designates the diameter, 2h the wall thickness and l the effective frame spacing:

$$p = \frac{E}{n^2 + \frac{\alpha^2}{2}} \frac{h}{a} \left[\frac{2\alpha^4}{(n^2 + \alpha^2)^2} + 0.73 \, (n^2 + \alpha^2)^2 \, h^2/a^2 \right] \quad \ldots \ldots \ldots \quad (8)$$

where $\quad \alpha = \pi a/l$

For n use that whole number which makes the expression for p a minimum for a given a, h, and l. E designates the elastic modulus of the material as long as the stresses in the shell are below the proportional limit. Concerning the applicability of the equations above the proportional limit, similar conditions apply as in the case of ordinary buckling of rods (see below).

Eq. (7) and (8) do not give accurate values for a tube of infinite length since here, as is well known, the number of lobes, which we have assumed to be large, decreases to two.

2. DISCUSSION of the FINAL FORMULA. TABLES.

Eq. (6), as well as the simplified Eq. (7), represents a straight line in an x – y coördinate system. Each wave number, n, represents a straight line for a given α/l, as shown in Z. Hence, y as a function of x will be represented by the ordinates of a straight line polygon. The vertexes of each one of these polygons increase in number towards the origin of the coördinate system.

The significance of α requires more detailed explanation. According to Z, Eq. (1.3),

$$\alpha = \pi a/l \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (9)$$

[illegible] the tube is to be substituted here for l; that is, the

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spacing of two successive nodal points of the lobe line appearing in a longitudinal section of the tube. If, as is the case in the fire tubes of a boiler, the tube ends can be regarded as fixed radially, then l can be taken equal to the total length between the ends of the tube. If the end supports are yielding, a somewhat higher value of l must be used. If the tube is divided into several belts by sufficiently rigid frames, the effective length l is an average value of the individual frame spacings. It is possible to obtain more exact data for determining the average value from the known calculated results for the buckling of rods with several bays. However, since the frames in submarine hulls are usually equally spaced, no difficulty is presented.

It is a question whether in many cases a higher number of waves between two successive frames, that is to say dividing l into halves, thirds, etc., may not result in a smaller value of the buckling pressure. This assumption seems not improbable when we consider that the smallest possible number of lobes, n = 2, in a circumferential belt between two frames, does not always result in the smallest critical pressure. In reality Eq. (7) indicates, in contrast to (C) and (D) of Z, that for fixed n and x the ordinate y does not increase unconditionally with α. For sufficiently small values of x, y decreases; for example, if α is made three times as large while n remains constant. However, it does not follow from this that the foregoing assumption is true since it is not yet certain whether a lower buckling pressure occurs by changing n while α is increased. The question to decide is whether the polygons that are constructed for a single α intersect in the region of very small x or whether the polygons always lie above each other and meet only in the origin. The answer is apparent if we consider that with small x on the one hand the number of lobes increases without limit, while on the other hand the vertexes of the polygons increase without limit. From the first condition it follows that Eq. (7), by expanding in powers of $1/n^2$ and neglecting the higher powers, can be simplified to:

$$y = \frac{(1-\sigma^2)\alpha^4}{n^6}\left(1 - \frac{5\alpha^2 - 2}{2n^2}\right) + n^2\left(1 + \frac{3\alpha^2 + 2}{2n^2}\right)x \quad . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

From the second condition it follows that we may with sufficient accuracy, in the neighborhood of the origin, consider the polygon as the envelope of the family of straight lines determined by Eq. (10) for successive values of n. This envelope is obtained through Eq. (10) in conjunction with the following equation obtained by the differentiation of Eq. (10) with respect to n^2:

$$0 = -\frac{3(1-\sigma^2)\alpha^4}{n^8}\left[1 - \frac{2(5\alpha^2 - 2)}{3n^2}\right] + x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (11)$$

Elimination of n^2 from Eq. (10) and (11) gives the equation of the envelope:

$$y = 4\sqrt[4]{\frac{1-\sigma^2}{27}}\,\alpha\sqrt[4]{x^3} + \frac{2}{3}(\alpha^2 + 2)x = 1.714\,\alpha\sqrt[4]{x^3} + \frac{2}{3}(\alpha^2 + 2)x \quad . . . (12)$$

As can be seen from Eq. (12), since the ordinates increase with increase of α, no intersection occurs, and we come to the conclusion: The minimum buckling pressure always corresponds to a deformation without nodal points between the points of support in the longitudinal plane, and, therefore, to the smallest possible value of α.

[Translators Note: Equation (12) derived to justify the conclusion in the preceding paragraph is a good approximation to the envelope of Eq. (7) in the neighborhood of the origin, that is, for small values of x and α. The exact equation of the envelope has a very important property: It can be used in place of the original equation (7) to determine y with sufficient accuracy for any given values of x and α. Graphically, this fact is evident from the proximity of the envelope and those parts of the family of straight lines used to determine y represented by Eq. (7) with n as a parameter. Analytically it is evident from the fact that the equation of the envelope, obtained by eliminating n between Eq. (7) and (11), is in effect Eq. (7) with that value of n substituted which for any given x and α makes y a minimum.

Eq. (12) is an approximate equation of the envelope of Eq. (7), valid only in the neighborhood of the origin, because of the nature of the assumptions which led to Eq. (10). The equation of the envelope of Eq. (7) should be derived directly from Eq. (7) itself. When so derived, its usefulness is no longer limited to the neighborhood of the origin. This equation can be very accurately derived as follows:

Differentiating Eq. (7) with respect to either n, $(n^2 + \alpha^2)$, or ρ and equating to zero we get

$$(n^2 + \alpha^2)^5 x - (n^2 + \alpha^2)^4 \alpha^2 x - 3(n^2 + \alpha^2)\alpha^4(1 - \sigma^2) + \alpha^6(1 - \sigma^2) = 0 \quad (11a)$$

The solution of (11a) for n gives that value of n which will make Eq. (7) a minimum for any given α and x. A very approximate solution can be readily obtained. Rewriting Eq. (11a)

$$(n^2 + \alpha^2)^4\left[(n^2 + \alpha^2)x - \alpha^2 x\right] - 3\alpha^4(1 - \sigma^2)\left[n^2 + \alpha^2 - \alpha^2/3\right] = 0$$

$$(n^2 + \alpha^2)^4 = \frac{3\alpha^4(1 - \sigma^2)(n^2 + 2\alpha^2/3)}{n^2 x}$$

$$n^2 + \alpha^2 = \alpha\sqrt[4]{\frac{3(1 - \sigma^2)}{x}}\sqrt[4]{1 + \frac{2}{3}\alpha^2/n^2} = \alpha\sqrt[4]{\frac{3(1 - \sigma^2)}{x}}\sqrt[4]{k/3} \quad . . . (11b)$$

[illegible] . . . . (11c)

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$$y = \frac{\frac{1+k}{k}\sqrt[4]{k(1-\sigma^2)}\,\alpha\,x^{\frac{3}{4}}}{1-\frac{\alpha x^{\frac{1}{4}}}{2\sqrt[4]{k(1-\sigma^2)}}} \quad \text{. . . . . (12a)}$$

In the majority of practical cases, the value of the ratio α/n lies between 2/3 and 1/3. Placing $\alpha/n = 1/2$ in Eq. (11c) and substituting (11c) in (12a) we obtain the simple expression

$$y = \frac{1.718\ \alpha\sqrt[4]{x^3}}{1-0.374\,\alpha\sqrt[4]{x}} \quad \text{. . . . . (a)}$$

With $\alpha/n = 2/3$ we obtain the slightly altered equation

$$y = \frac{1.724\ \alpha\sqrt[4]{x^3}}{1-0.365\,\alpha\sqrt[4]{x}} \quad \text{. . . . . (b)}$$

while $\alpha/n = 1/3$ gives

$$y = \frac{1.715\ \alpha\sqrt[4]{x^3}}{1-0.382\,\alpha\sqrt[4]{x}} \quad \text{. . . . . (c)}$$

Any number of similar equations may be obtained by using different values of the ratio α/n. As a limiting case $\alpha/n = 0$ gives the equation

$$y = \frac{1.714\ \alpha\sqrt[4]{x^3}}{1-0.389\,\alpha\sqrt[4]{x}} \quad \text{. . . . . (d)}$$

which differs but slightly from (c).

If equation (d) be expanded in a series, the first two terms are identical with the first two terms of equation (12). Moreover, the expansion of any of the equations (a) to (d) yields a series all terms of which are positive. Therefore, the same conclusion can be drawn from these equations as from Eq. (12) concerning the impossibility of nodal points between points of support in the longitudinal plane. It is evident that Eq. (a) – (c) are much better approximations to the envelope than Eq. (12). Each gives extremely accurate values in that particular region of α/n for which it was developed and good values for a considerable range of α/n.

To determine just how closely these approximate equations check with Eq. (6), about 70 values of y, covering the region $x = 4/3 \cdot 10^{-6}$ to $12 \cdot 10^{-6}$ ($t/d = 0.002$ to 0.006) and $\alpha = 1/4$ to 16 ($l/d = 6.28$ to 0.1), were computed by each of the six equations (6), (7), (a), (b), (c), and (d). These values and their percentage deviations from the exact values computed by Eq. (6) are given in Tables II and III.

Of the four approximate equations tried, equation (a) shows the best agreement with the exact equation (6). The agreement is excellent for all values of l/d less than 2. The mean of the absolute deviations throughout the range of l/d = 0.1 to 2 is 0.5 per cent. The maximum discrepancy is less than 3 per cent for l/d = 2 and is considerably below 2 per cent for all values of l/d equal to or less than 1.6 and is less than 1 per cent for all values of l/d less than 1. In fact, Eq. (a) gives better agreement with Eq. (6) than does Eq. (7) from which it was derived!

Equation (a) can be put in a more convenient form by substituting values of y, x, and α from Eq. (6') and (9), whence

$$p = \frac{\frac{2.424\,E}{(1-\sigma^2)^{3/4}}\,(h/a)^{5/2}}{\frac{l}{2a} - 0.447\sqrt{h/a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a')$$

$$\text{or } (\sigma = 0.3) \quad p = \frac{2.60\,E\,(t/d)^{5/2}}{\frac{l}{d} - 0.45\sqrt{t/d}}$$

Again it might be pointed out that the usefulness of Eq. (a') is not confined to the neighborhood of the origin. It is a simple equation, independent of n, which determines p with a high degree of accuracy for a large range of values of l/d and t/d. Equation (a') may replace equation (6) in all practical computations where l/d is not greater than 2. When l/d is less than 1 the discrepancy will be less than 1 per cent.

In order to facilitate the practical application of the results derived above, one may use to advantage a graph, Fig. 4, that contains the polygons (they appear as curves) for the values α = 2, 4, 6, ... to 20, within the range up to x = 6 x 10^{-6} corresponding to the ratio a/l from approximately 2/3 to 6 (l/d = 0.8 to 0.08) and to the ratio h/a up to about 0.0045. The straight lines are computed throughout by the complete equation (6). In the neighborhood of the origin, the polygons are made up of curves determined by Eq. (12). The corresponding angles of the polygons are connected by dotted lines, forming quadrangular areas for the various wave numbers. The separate vertical straight lines correspond to the constant values of 1000 h/a written beneath; the values of the ratios πa/l are written on the sides of the polygons. On the axis of ordinates, a second scale is represented beside the y-scale which, under the assumption of an elastic modulus E = 2,125,000 kg/cm² (30.22 x 10^6 lb. per sq. in.), gives the value of

$$k = p\,a/2h = \frac{E}{1-\sigma^2}\,y = 2.34 \times 10^6\,y \quad \ldots\ldots\ldots\ldots (13)$$

[illegible] kg/cm² (27,[illegible] lb. per sq. in.) The values in the range of [illegible]

[illegible] to the observations of Tokugawa on [illegible]

buckling of rods. According to these, as is well known, the critical pressure for a rod of slenderness ratio r/l is determined by means of two constants c and K by the formula

$$k \;=\; K(1 - c\, l/r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

where for mild steel $K = 3100$ kg/cm² (44,000 lb. per sq. in.) and $c = 0.00368$.

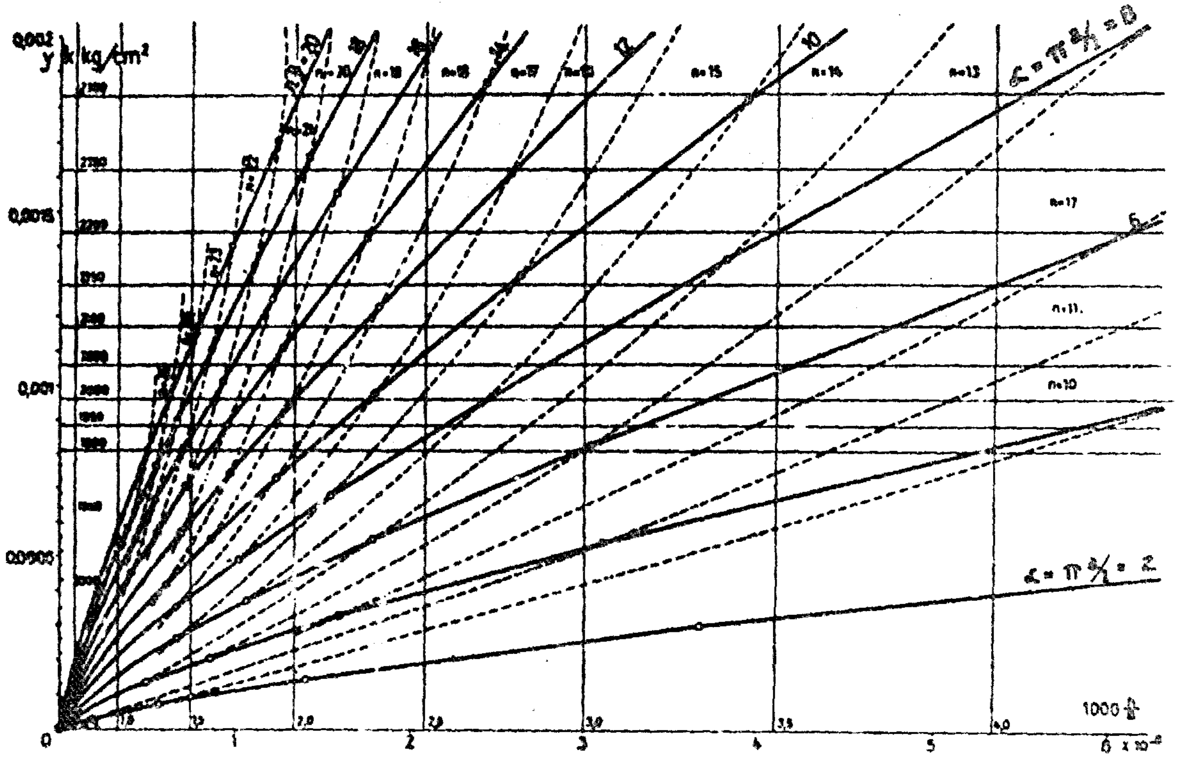

FIG. 4. GRAPH FOR DETERMINATION OF BUCKLING PRESSURE.

One might so interpret Eq. (14), as if in place of the actual elastic [illegible], E, the expression

$$E \;[illegible]\; (1 - k/K)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1[illegible])$$

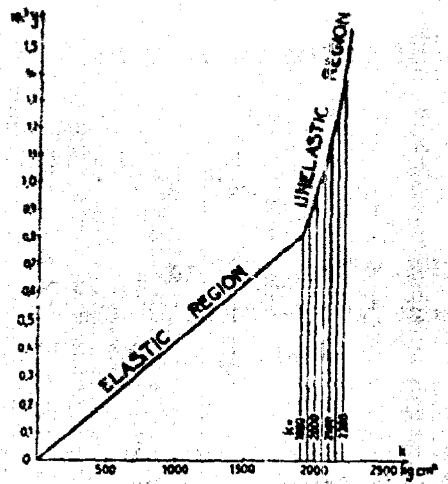

FIG. 5. Relationship between K and y.

were to be substituted in the otherwise unaltered Euler buckling formula. We have employed this expression for E in Eq. (13) for the conversion of y into k for $k \geq 1900$, $y \geq 0.000809$, and thereby obtained the k-scale for the unelastic region. The relation between y and k is represented in Fig. 5.

The application of the graphs is as follows. For the given value of the ratio, wall thickness to diameter, i.e. h/a, we seek the corresponding ordinate, as well as the corresponding polygon, for the given ratio of the radius to effective length a/l. The ordinate of the intersection of these two lines gives the value of y or of k from which the buckling pressure p is obtained.

$$p = \frac{E}{1-\sigma^2}\,\frac{2h}{a}\,y, \qquad \text{or} \quad p = \frac{2h}{a}\,k \quad \ldots\ldots\ldots\ldots (16)$$

If no lines exist for the exact ratio given, y can be determined by interpolation. In any event, the graph indicates the number of lobes, n, which is determinative for the given case. For a known n, it is then not difficult to compute the exact value of y by equations (6), (7), or (8).

The following numerical table was obtained by readings from the graph. It is valid both below and above the proportional limit, providing the cylindrical shell is manufactured from medium steel plate.

Table I.

Critical pressure p in kg/cm² for various wall-thickness ratios, h/a, and length ratios, a/l.

$\alpha = \frac{\pi a}{l}$	a/l	1000 h/a							
		0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
2	0.637	0.045	0.23	0.6	1.3	2.2	3.6	5.3	7.4
4	1.273	0.080	0.46	1.3	2.6	4.6	7.4	10.9	15.3
6	1.910	0.125	0.72	2.0	4.0	7.2	11.5	14.2	17.2
8	2.546	0.175	1.0	2.7	[illegible]	9.6	12.5	15.4	18.3
10	3.183	0.22	1.2	3.5	7.2	10.3	13.2	16.1	
12	3.820	0.26	[illegible]	4.2	8.0	10.9	13.8		
14	4.456	0.31	1.8	5.1	8.4	11.2			
16	5.093	0.37	2.1	5.8	8.7	11.6			
[illegible]	[illegible]	[illegible]	2.4	6.0	9.0				
[illegible]	[illegible]	[illegible]	2.8	6.2	9.2				

If we proceed purely analytically, without the use of tables, we must use the equation

$$p = \frac{2h}{a} K \left[1 - \frac{\sqrt{1-\sigma^2}\,\gamma_c}{\sqrt{y}} \right] = \frac{h}{a}\, 6200 \left[1 - \frac{0.011}{\sqrt{y}} \right] \quad \ldots \ldots \ldots \ldots \ldots (17)$$

in the unelastic region, which follows from the substitution of Eq. (15) in Eq. (13) and the elimination of k. The value of y must here be determined by the equations in Section 1. The numerical values in Eq. (17) are for medium steel.

3. COMPARISON with EXPERIMENTAL RESULTS.

Two series of experiments were carried out in 1918 with large medium steel tubes welded along the longitudinal seam, and subjected to external water pressure. The first series included tubes with the wall thickness-ratio

$$h/a = 1/400 = 0.0025; \qquad x = h^2/3a^2 = 2.083 \times 10^{-6}$$

and the three length ratios

$$a/l = 400/120,\ 400/180,\ 400/240 \text{ whence,}$$

$$\alpha = \pi a/l = 10.47,\ 6.98,\ 5.24$$

We obtain from the graph, Fig. 4, at the ordinate h/a = 0.0025, the values of the number of lobes, n, in the three cases:

(a) n = 16 (b) n = 14 (c) n = 13

[Translator's Note: For case (c), n lies so close to the border line between 12 and 13 that practically it might be either. Actually, n = 12 is determinative.]

The experimental number of lobes from the tests on four different sizes of models of diameters 800, 1200, 1600, 2400 mm (31.5 in., 47.2 in., 63.0 in., 94.5 in.) are

Model Size		Case (a)	(b)	(c)
I	(D = 800 mm)	n = 17	n = 15	n = 14
II	(D = 1200 mm)	17	15	13 – 14
III	(D = 1600 mm)	—	14 – 15	–
IV	(D = 2400 mm)	16	14 – 15	–

We may conclude, therefore, that the agreement is almost exact. In addition it is worthy of note that the agreement appears to improve for the larger models. The differences between the number of lobes observed in the various sizes of models, as well as the departures from the calculated number, hardly exceed the limits of unavoidable experimental error. Fig. 6 shows the buckled model in [illegible] IVa.

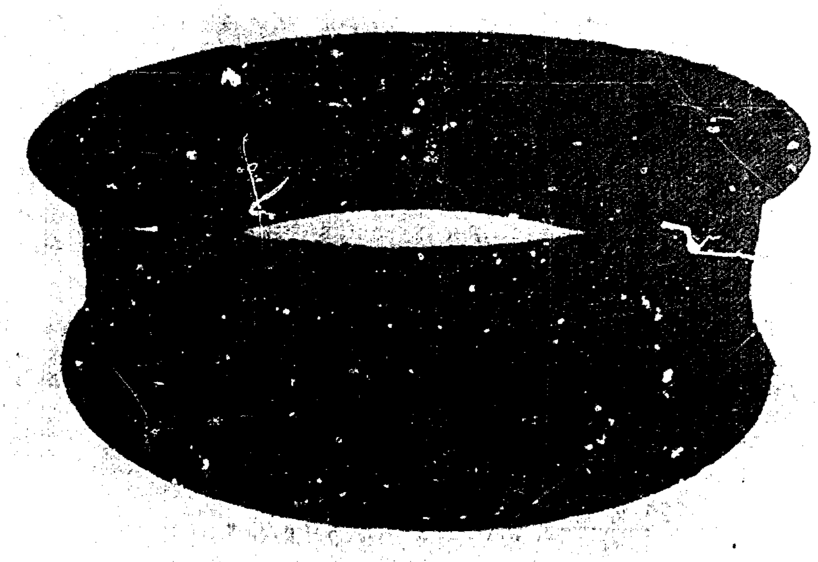

FIG. 6. A Portion of the Experimental Tube IVa with 16 lobes Around the Circumference.

The situation is quite different in determining the buckling pressure itself. Here the experimental values obtained for the individual sizes of models differ very materially from each other. Since the collapsing pressures increase directly as the size of the model, it might be expected that the calculated values should correspond to a limiting case of an infinitely large test body. According to Eq. (6), using the approximation of Fig. 3, the computations show:

[Translator's Note: Numerous minor numerical errors, probably due to the use of a slide rule, have been corrected by the translator without specific reference in each case. Changes from the original text are indicated * .]

Case (a) $\alpha = 10.47^*$ $\quad n = 16$ $\quad \rho = \dfrac{\alpha^2}{\alpha^2 + n^2} = 0.30$

$\mu_1 = 1.53 \qquad \mu_2 = 1.23$

$$y = \frac{0.91 \times 0.09}{309.8^*} + \frac{2.083}{309.8^*}\left(365.6^{2*} - 2 \times 1.53 \times 256 + 1.23\right) \times 10^{-6}$$

$$y = 0.000264^* \div C.000894^* = 0.00116$$

[illegible] this value lies above the limit 0.00081, p must be determined from y by [illegible].

$$[illegible] \times 4000 \left(1 - \frac{0.011^*}{0.0081}\right) = 10.5^* \text{ kg/cm}^2 = 10.2^* \text{ atm.}$$

[Translator's Note: Substituting the above values $\alpha = 10.47$, $x = 2.083 \times 10^{-6}$ in (a) gives

$$y = \frac{1.718 \times 10.47 \times (2.083 \times 10^{-6})^{3/4}}{1 - 0.374 \times 10.47 \times (2.083 \times 10^{-6})^{1/2}} = 0.00116 \Big]$$

Case (b) $\alpha = 6.98^*$ $n = 14$ $\rho = 0.199$ $\mu_1 = 1.35$ $\mu_2 = 1.20$

$$y = \frac{0.91 \times 0.0396^*}{219.4} + \frac{2.083}{219.4}(244.7^2 - 2 \times 1.35 \times 196 + 1.20) \times 10^{-6}$$

$$y = 0.000164 + 0.000564 = 0.00073$$

This value lies below the limit and therefore according to Eq. (13)

$$p = 0.005\,\frac{2.125 \times 10^6}{0.91} \times 0.00073 = 3.5\ \text{kg/cm}^2 = 8.2^*\ \text{atm.}$$

[Translator's Note: Substituting $E = 2.125 \times 10^6$ kg. per sq. cm.

$t/d = 0.0025$ $\quad l/d = \dfrac{180}{2 \times 400} = 0.225$ directly in Eq. (a')

$$p = \frac{2.60 \times 2.125 \times 10^6 \times (0.0025)^{5/4}}{0.225 - 0.45 \times \sqrt{0.0025}} = 8.53\ \text{kg/cm}^2 = 8.26\ \text{atm.} \Big]$$

Case (c). $\alpha = 5.24$, $n = 13$, $\rho = 0.140$, $\mu_1 = 1.24^*$, $\mu_2 = 1.2$

$$y = \frac{0.91 \times 0.0196}{181.7} + \frac{2.083}{181.7}(196.5^{2*} - 2 \times 1.24^* \times 169 + 1.2) \times 10^{-6}$$

$$= 0.000098 + 0.000438 = 0.000536.$$

$$p = 0.005\,\frac{2.125 \times 10^6}{0.91} \times 0.000536 = 6.26^*\ \text{kg. per sq.cm.} = 6.06^*\ \text{atm.}$$

[Translator's Note: The use of $n = 12$ instead of $n = 13$ in the above problem gives a slightly smaller value of y.

$\alpha = 5.24$, $n = 12$, $\rho = 0.160$, $\mu_1 = 1.28$ $\mu_2 = 1.2$

$$y = \frac{0.91 \times 0.0256}{156.7} + \frac{2.083}{156.7}(171.5^2 - 2 \times 1.28 \times 144 + 1.2) \times 10^{-6}$$

$$= 0.000149 + 0.000386 = 0.000535$$

However, in practice, collapse may occur in either 12 or 13 lobes. The use of Eq. (a') gives

$$[illegible] 2.125 \times 10^6 \times (0.0025)^{5/4} / [illegible] \sqrt{0.0025} = 6.22\ \text{kg. per sq.cm.} = 6.02\ \text{atm.} \Big]$$

[illegible] values [illegible]

		(a)	(b)	(c)
Theoretical Values:		$p = 10.2$*	$p = 8.2$*	$p = 6.1$*
Experimental Values:		(a)	(b)	(c)
Model Size	I	6.1	3.9	3.35
	II	7.0	5.1	4.15
	III	—	6.0	—
	IV	9.5	7.2	—

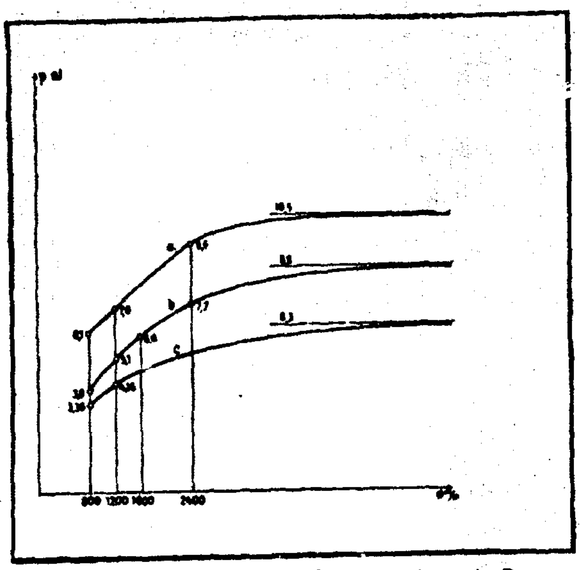

FIG. 7. Summary of Experimental Data.

The entirely disproportionate value of IIIa is omitted here. Fig. 7 gives a general idea of the distribution of these values. Here we see that the calculated values may very well be regarded as obtainable asymptotic limits. There remains only the question as to what causes the smaller models to collapse so far below the theoretical collapsing pressure.* It seems most probable that the welding of the thin plating disturbs the symmetry of the circular form, or the homogeneity of the material itself. It is certain that in the case of buckling with higher number of lobes, even very slight irregularities may have a decisive effect.

The second group of experiments concerns tubes with the wall thickness-ratio

$$h/a = 0.00406, \qquad x = h^2/3a^2 = 5.49^* \times 10^{-6}$$

and with frame spacing of 400, 500, and 600 mm (15.7 in., 19.7 in., 23.6 in.) and a tube radius of 800 mm (31.5 in.). In both the first two cases the frames failed under the test. In the third case, likewise, it must be assumed as certain, because of the magnitude of the observed buckling pressure, that the limit of strength was reached through failure of the frame.

[Translator's Note: It is understood that the testing arrangements were such as to make it impossible to observe the inside of the model during the tests.]

As we know from a theory which will not be further discussed here, if the frames are too weak, the buckling pressure must correspond to that of a tube with an effective length of double the frame spacing or even higher. We can, therefore, test the reliability of our theory by computing the buckling pressure for tubes of the length $l = 800$, 1000 and 1200 mm and compare them with the test results. With help of the graph we find:

(1) $\alpha = 3.14$, $n = 8$ (2) $\alpha = 2.51$, $n = 8$ (3) $\alpha = 2.09$, $n = 7$

[illegible] calculation shows:

[illegible] 0.134 $\mu_1 = 1.24$ $\mu_2 = 1.2$

$$y = \frac{0.91 \times 0.018}{67.95} + \frac{5.49^*}{67.95}(73.9^2 - 2 \times 1.24 \times 64 + 1.2) \times 10^{-6}$$

$$= 0.000241 + 0.000428^* = 0.000669^*$$

$$p = 0.00812 \times \frac{2.125 \times 10^6}{0.91} \times 0.000669^* = 12.7^* \text{ kg. per sq.cm.} = 12.3^* \text{ atm.}$$

[Translator's Note: This value of p is too high since n = 9 is determinative instead of n = 8. Using n = 9, $\rho = 0.109$, $\mu_1 = 1.19$, $\mu_2 = 1.1$

$$y = \frac{0.91 \times 0.0119}{84.95} + \frac{5.49}{84.95}(90.9^2 - 2 \times 1.19 \times 81 + 1.1) \times 10^{-6}$$

$$= 0.000127 + 0.000522 = 0.000649$$

$$p = 0.00812 \times \frac{2.125 \times 10^6}{0.91} \times 0.000649 = 12.31 \text{ kg. per sq.cm.} = 11.9 \text{ atm.}$$

The use of equation (a') gives

$$p = \frac{2.60 \times 2.125 \times 10^6 \times (0.00406)^{5/2}}{0.5 - 0.45\sqrt{0.00406}} = 12.31 \text{ kg. per sq. cm.} = 11.9 \text{ atm.}$$

It will be noted that Eq. (a') gives the correct collapsing pressure independent of the number of lobes, while the use of Eq. (6) with an incorrect value of n gives a collapsing pressure which is too high.]

(2) $\rho = 0.0896^*$ $\mu_1 = 1.16$ $\mu_2 = 1.1$

$$y = \frac{0.91 \times 0.00803^*}{66.15} + \frac{5.49^*}{66.15}(70.3^{2*} - 2 \times 1.16 \times 64 + 1.1) \times 10^{-6}$$

$$= 0.000110^* + 0.000398^* = 0.000508^*$$

$$p = 0.00812 \times \frac{2.125 \times 10^6}{0.91} \times 0.000508^* = 9.63^* \text{ kg. per sq.cm.} = 9.3^* \text{ atm.}$$

[Translator's Note: The use of Eq. (a') gives:

$$p = \frac{2.60 \times 2.125 \times 10^6 \times (0.00406)^{5/2}}{0.625 - 0.45\sqrt{0.00406}} = 9.73 \text{ kg. per sq.cm.} = 9.4 \text{ atm.}$$]

(3) $\rho = 0.0820$ $\mu_1 = 1.13$ $\mu_2 = 1.1$

$$y = \frac{0.91 \times 0.0067}{50.2} + \frac{5.49^*}{50.2}(53.4^2 - 2 \times 1.13 \times 49 + 1.1) \times 10^{-6}$$

$$= 0.000121 + 0.000300^* = 0.000421^*$$

$$p = 0.00812 \times \frac{2.125 \times 10^6}{0.91} \times 0.000421^* = 7.98^* \text{ kg. per sq.cm.} = 7.7^* \text{ atm.}$$

[Translator's Note: The use of Eq. (a') gives:

$$p = \frac{2.60 \times 2.125 \times 10^6 \times (0.00406)^{5/2}}{0.750 - 0.45 \times \sqrt{0.00406}} = 8.04 \text{ kg. per sq.cm.} = 7.8 \text{ atm.}]$$

The experimental pressures 12.2, 9.2, and 8.5 atmospheres are to be compared with the three calculated values thus obtained, 11.9, 9.3, and 7.7 atmospheres. [Translator's Note: These values are given as 12.8, 9.5, and 8.1 in the text.] We see that there is good agreement.

The number of lobes, n = 14 – 15, observed in the third test, as compared to the theoretical number, n = 7, cannot be explained. Probably it is due to an error made by the observer. In both the other tests it was impossible to determine the number of lobes after failure.

Conclusions (added by translator).

Equation (6) gives a solution for the collapsing pressure of a circular cylindrical vessel closed at the ends by flat heads and subjected to uniform hydrostatic pressure on both shell and heads. As will be observed, the determination of the correct collapsing pressure depends not only upon the length, diameter and thickness, but also upon the number of lobes, n, into which a circumferential belt of the vessel, between stiffening rings, divides itself at collapse. Equation (6) gives a different value of p for each assumed value of n and collapse occurs in that number of lobes for which p is a minimum. The use of Eq. (6), therefore, involves the determination of the value of n which gives a minimum value of p, either by actual substitution in that equation or by the use of curves previously prepared, such as given in Fig. 4. However, the determination of n by Fig. 4 is not always accurate, as was shown in the solution of example 1, page 15, where the author by using n = 8 instead of the correct value n = 9 obtained a collapsing pressure which was considerably too high. Fig. 8 on page 18 is drawn to a different scale and gives a very accurate determination of n.

Equation (7) is simpler than Eq. (6) and the collapsing pressures calculated by it differ from those calculated by Eq. (6) by less than 1 per cent for all values of l/d below 0.5, as shown by Tables II and III, page 19. However, Eq. (7) likewise requires the determination of that value of n which gives the minimum collapsing pressure and hence is somewhat cumbersome and indirect.

Equation (a) presents a striking contrast to equations (6) and (7). It is very simple and gives the collapsing pressure directly without the use of n. Moreover, Eq. (a) checks Eq. (6) even more closely than does Eq. (7), and can replace Eq. (6) in all practical calculations.

NUMBER OF LOBES "n" INTO WHICH A TUBE WILL COLLAPSE WHEN SUBJECTED TO UNIFORM RADIAL AND AXIAL EXTERNAL PRESSURE

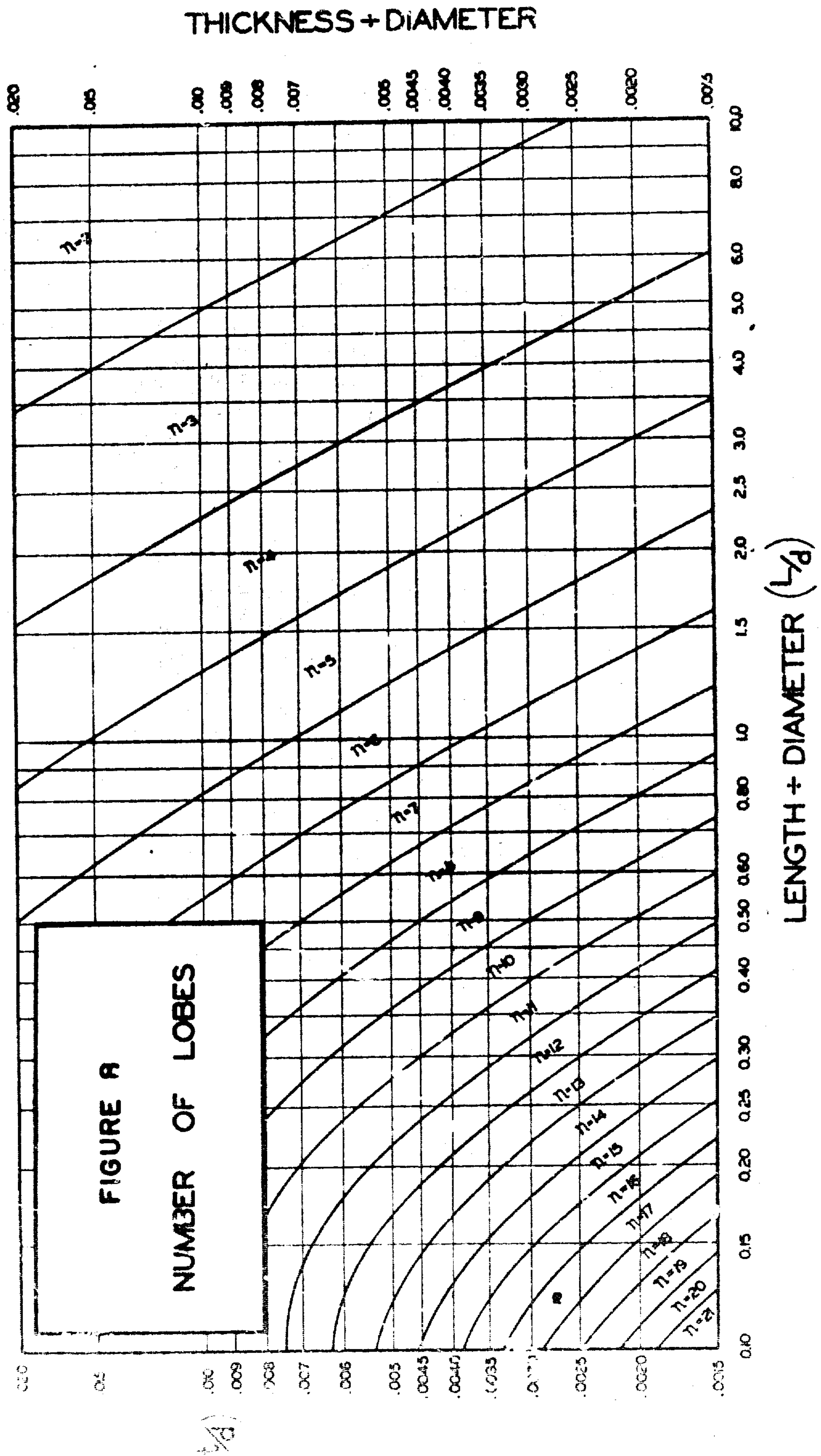

FIGURE R

NUMBER OF LOBES

Best Available Copy

	[illegible] 10^{-6}					t/d = 0.002	x = 3 x 10^{-6}					t/d = 0.003	x = 16/3 x 10^{-6}					t/d = 0.004	x = 25/3 x 10^{-6}					t/d = 0.005	x = 12 x 10^{-6}					t/d = 0.006
	10^4 y computed by Eq.						10^4 y computed by Eq.						10^4 y computed by Eq.						10^4 y computed by Eq.						10^4 y computed by Eq.					
	6	7	a	b	c	d	6	7	a	b	c	d	6	7	a	b	c	d	6	7	a	b	c	d	6	7	a	b	c	d
[illegible]	[illegible]	[illegible]	0.163	0.170	0.167	0.169	0.297	0.321	0.311	0.312	0.310	0.310	0.485	0.533	0.479	0.480	0.478	0.478	0.728	0.806	0.669	0.672	0.668	0.668	1.024	1.139	0.881	0.884	0.879	0.879
[illegible]	[illegible]	0.332	0.339	0.340	0.339	0.338	0.603	0.625	0.624	0.626	0.623	0.623	0.961	1.007	0.962	0.965	0.960	0.960	1.422	1.498	1.345	1.351	1.344	1.343	1.820	1.854	1.772	1.777	1.768	1.767
[illegible]	[illegible]	0.534	0.535	0.537	0.534	0.534	0.964	0.987	0.984	0.988	0.983	0.983	1.545	1.592	1.518	1.524	1.517	1.516	2.143	2.183	2.126	2.133	2.123	2.122	2.726	2.801	2.799	2.809	2.796	2.794
[illegible]	[illegible]	0.682	0.683	0.685	0.682	0.681	1.282	1.307	1.258	1.262	1.256	1.256	1.907	1.942	1.941	1.948	1.939	1.938	2.671	2.737	2.719	2.728	2.717	2.715	3.605	3.709	3.581	3.593	3.578	3.576
[illegible]	[illegible]	1.295	1.380	1.384	1.379	1.378	1.975	1.998	1.993	2.000	1.792	1.991	3.095	3.125	3.061	3.091	3.078	3.078	4.255	4.318	4.321	4.335	4.317	4.317	5.673	5.775	5.698	5.715	5.695	5.693
[illegible]	[illegible]	1.403	1.383	1.388	1.382	1.381	2.554	2.579	2.555	2.561	2.553	2.553	3.917	3.954	3.954	3.966	3.951	3.951	5.536	5.606	5.550	5.566	5.548	5.547	7.379	7.447	7.327	7.348	7.325	[illegible]
[illegible]	[illegible]	2.113	2.102	2.108	2.101	2.100	3.886	3.906	3.896	3.907	3.894	3.895	6.071	6.117	6.044	6.059	6.042	6.044	8.438	8.503	8.504	8.524	8.505	8.507	11.250	11.355	11.250	11.275	11.253	11.257
[illegible]	2.813	2.844	2.841	2.843	2.839	2.840	5.301	5.327	5.280	5.292	5.279	5.283	8.201	8.247	8.214	8.231	8.216	8.222	11.526	11.590	11.587	11.608	11.594	11.602	15.360	15.467	15.357	15.381	15.369	15.382
[illegible]	[illegible]	3.598	3.599	3.607	3.598	3.600	6.721	6.747	6.712	6.725	6.714	6.729	10.467	10.513	10.471	10.486	10.477	10.489	14.744	14.810	14.809	14.827	14.826	14.843	19.750	19.832	19.673	19.689	19.700	19.726
[illegible]	4.367	4.378	4.378	4.386	4.379	4.392	8.216	8.243	8.194	8.205	8.230	8.210	12.862	12.900	12.817	12.829	12.832	12.851	18.148	18.220	18.173	18.181	18.232	18.231	24.120	24.213	24.208	24.210	24.255	24.301
[illegible]	5.167	5.178	5.180	5.186	5.182	5.187	9.722	9.745	9.725	9.733	9.735	9.750	15.230	15.271	15.261	15.266	15.285	15.315	21.692	21.747	21.700	21.696	21.748	21.793						
[illegible]	5.993	6.005	6.003	6.009	6.008	6.016	11.290	11.314	11.314	11.317	11.330	11.352	17.777	17.821	17.807	17.802	17.844	17.587	25.313	25.390	25.397	25.374	25.468	25.535						
[illegible]	7.716	7.726	7.723	7.724	7.734	7.749	14.633	14.656	14.662	14.651	14.697	14.737	23.173	23.215	23.244	23.297	23.318	23.396												
[illegible]	9.541	9.550	9.546	9.538	9.566	9.591	18.208	18.229	18.267	18.232	18.328	18.393																		
[illegible]	11.458	11.46[illegible]	11.482	11.462	11.515	11.553	22.053	22.075	22.168	22.088	22.2.6	22.365																		
[illegible]	13.482	13.491	13.541	13.504	13.592	13.646	26.168	26.184	26.383	26.264	26 530	26.674																		

TABLE II

Values of y computed by Equations (6), (7), (a), (b), (c), (d)

		x = 4/3 x 10^{-6}				t/d = 0.002	x = 3 x 10^{-6}				t/d = 0.003	x = 16/3 x 10^{-6}				t/d = 0.004	x = 25/3 x 10^{-6}				t/d = 0.005	x = 12 x 10^{-6}				t/d = 0.006
		Percentage Variation by Eq.					Percentage Variation by Eq.					Percentage Variation by Eq.					Percentage Variation by Eq.					Percentage Variation by Eq.				
[illegible]	[illegible]	7	a	b	c	d	7	a	b	c	d	7	a	b	c	d	7	a	b	c	d	7	a	b	c	d
1/4	6.28	4.32	4.32	4.94	4.32	4.32	8.08	4.71	5.05	4.38	4.38	9.90	-1.24	-1.09	-1.44	-1.44	12.71	-8.19	-7.69	-8.24	-8.24	11.23	-13.96	-13.67	-14.16	[illegible]
1/2	3.14	1.44	-2.3[illegible]	2.02	-2.31	-2.59	3.65	3.48	3.81	3.32	3.32	4.79	0.10	0.42	-0.10	-0.10	5.34	-5.41	-4.99	-5.49	-5.56	1.87	-2.64	-2.36	-2.86	-2.91
3/4	2.09	0.91	-2.[illegible]	-2.19	-2.73	-2.73	2.39	2.08	2.49	1.97	1.97	3.34	-1.75	-1.36	-1.81	-1.88	1.73	-0.79	-0.47	-0.93	-0.98	2.75	2.68	3.05	2.57	2.50
1	1.57	1.19	1.34	1.63	1.19	1.04	1.95	-1.87	-1.56	-2.03	-2.03	1.8[illegible]	1.78	2.15	1.68	1.63	2.47	1.80	2.13	1.72	1.65	2.88	-0.67	-0.33	-0.75	-0.80
[illegible]	[illegible]	1.01	-0.37	0	-0.46	-0.55	1.16	0.91	1.27	0.86	0.81	[illegible].97	-0.45	-0.13	-0.55	-0.55	1.48	1.55	1.88	1.46	1.46	1.80	0.44	0.74	0.39	0.35
2	0.79	0.[illegible]9	-0.65	-0.29	-0.72	-0.79	0.98	0.04	0.27	-0.04	-0.04	0.94	0.94	1.25	0.87	0.87	1.26	0.25	0.54	0.22	0.20	0.92	-0.70	-0.42	-0.73	-0.75
3	0.52	0.43	0.05	0.33	0.00	-0.05	0.51	0.[illegible]	0.54	0.21	0.23	0.76	-0.44	-0.18	-0.48	-0.44	0.77	0.78	1.02	0.79	0.82	0.93	0.09	0.22	0.03	0.36
4	[illegible]	0.39	0.28	0.53	0.21	0.25	0.49	-0.40	-0.17	-0.42	-0.34	0.56	0.16	0.37	0.18	0.26	0.56	0.53	0.71	0.59	0.66	0.73	0.02	0.14	0.06	0.14
5	0.31	0.31	0.33	0.56	0.31	0.36	0.39	-0.13	0.03	-0.10	-0.01	0.44	0.04	0.18	0.10	0.21	0.45	0.44	0.56	0.56	0.67	0.42	-0.39	-0.31	-0.25	-0.12
6	0.26	0.25	0.25	0.44	0.27	0.34	0.33	-0.27	-0.13	-0.19	-0.07	0.30	-0.35	-0.26	-0.23	-0.09	0.40	0.14	0.18	0.30	0.46	0.39	0.36	0.33	0.56	0.75
7	0.22	0.21	0.25	0.37	0.29	0.39	0.24	0.03	0.11	0.13	0.29	0.27	0.20	0.24	0.36	0.56	0.25	0.04	0.02	0.26	0.47					
8	0.20	0.20	0.17	0.27	0.25	0.38	0.21	0.21	0.24	0.35	0.55	0.25	0.17	0.14	0.38	0.62	0.26	0.33	0.24	0.61	0.88					
10	0.16	0.13	0.09	0.10	0.23	0.43	0.15	0.20	0.12	0.44	0.71	0.18	0.31	0.15	0.63	0.96										
12	0.13	0.09	0.05	-0.03	0.26	0.52	0.12	0.32	0.13	0.66	1.02															
14	0.11	0.08	0.21	0.03	0.50	0.83	0.10	0.52	0.16	0.97	1.41															
16	0.10	0.07	0.44	0.16	0.82	1.22	0.06	0.82	0.37	1.38	1.93															

TABLE III

Percentage variation of y computed by Equations (7), (a), (b) .c), (d)
Above the value computed by Equation (6)